

Abstract

We study observables and deformations of generalized Chern-Simons action and show how to apply these results to maximally supersymmetric gauge theories. We describe a construction of large class of deformations based on some results on the cohomology of super Lie algebras proved in the Appendix.

Generalized Chern-Simons action and maximally supersymmetric gauge theories

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1 Chern-Simons functional.

One can construct Chern-Simons functional for every differential associative \mathbf{Z}_2 -graded algebra \mathcal{A} equipped with trace tr . By definition trace is a linear functional vanishing on supercommutators; we assume that the trace is closed (vanishes on elements of the form da). This linear functional can be odd or even; we are mostly interested in the case when it is odd.

We can define invariant inner product in terms of trace: $\langle a, b \rangle = \text{tr} ab$. Let us assume that this inner product is non-degenerate.

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For every N we define the associative algebra \mathcal{A}_N as tensor product $\mathcal{A} \otimes \text{Mat}_N$ where Mat_N stands for the matrix algebra. (In other words, \mathcal{A}_N is an algebra of $N \times N$ matrices with entries from \mathcal{A}_N .) Using the trace in \mathcal{A} and the conventional matrix trace we define the trace and invariant inner product in \mathcal{A}_N .

Chern-Simons functional on $\Pi\mathcal{A}_N$ is defined by the formula

$$CS_N(A) = \frac{1}{2}\text{tr}AdA + \frac{2}{3}\text{tr}A^3 = \frac{1}{2}\text{tr}AdA + \frac{1}{3}\text{tr}A[A, A]$$

Here Π stands for the parity reversal. Chern-Simons functional is even if the trace has odd parity, as we will assume in what follows. We omit the index N in the notation CS_N .

Notice that in this definition we need really only the super Lie algebra structure and invariant inner product in the algebra \mathcal{A}_N :

$$CS(A) = \frac{1}{2} \langle A, dA \rangle + \frac{1}{3} \langle A, [A, A] \rangle .$$

The functional CS coincides with the standard Chern-Simons functional in the case when \mathcal{A} is the algebra $\Omega(M)$ of differential forms on three-dimensional manifold M equipped with a trace $\text{tr}C = \int_M C$.

An odd non-degenerate inner product on vector space induces such a product on its dual that can be regarded as an odd symplectic form. This form $\omega = dz^a \omega_{ab} dz^b$ can be used to define an odd Poisson bracket on functions: $\{F, H\} = \overleftarrow{\partial}_a F \omega^{ab} \overrightarrow{\partial}_b H$, and to assign a vector field $\partial_b F \omega^{ab}$ to every function. (This vector field corresponds to the first order differential operator ξ_F defined by the formula $\xi_F(H) = \{F, H\}$.)

Applying this remark to \mathcal{A}_N we see that the odd vector field Q corresponding to the functional CS can be written in the form

$$\delta_Q A = dA + \frac{1}{2}[A, A]. \tag{1}$$

This vector field obeys $[Q, Q] = 0$.¹ The relation $[Q, Q] = 0$ is equivalent to the BV master

¹An odd vector field Q obeying $[Q, Q] = 0$ is called homological vector field, because the corresponding first order differential operator \hat{Q} obeys $\hat{Q}^2 = 0$ and therefore can be considered as a differential acting on the space of functions.

equation $\{CS, CS\} = 0$. Notice that that vector field Q does depend on super Lie algebra $L_N \mathcal{A} = Lie \mathcal{A}_N$ corresponding to the associative algebra \mathcal{A}_N , but it does not depend of inner product.

The construction of the functional CS can be generalized to the case when \mathcal{A} is an A_∞ -algebra equipped with invariant inner product. Recall that the structure of A_∞ -algebra \mathcal{A} on a \mathbb{Z}_2 -graded space is specified by means of a sequence $^{(k)}m$ of operations, satisfying some relations that will be described later. The operation $^{(k)}m$ has k arguments, in a coordinate system it is specified by a tensor $^{(k)}m_{a_1, \dots, a_k}^a$ having one upper index and k lower indices. Having an inner product we can lower the upper index; invariance of inner product means that the tensor $^{(k)}\mu_{a_0, a_1, \dots, a_k} = \omega_{a_0 a} m_{a_1, \dots, a_k}^a$ is cyclically symmetric (in graded sense). The Chern-Simons functional can be defined on $\mathcal{A} \otimes \text{Mat}_N$ in the following way. In a basis a_1, \dots, a_k for \mathcal{A} the value of the functional on even $a = \sum a_i t^i, t^i \in \text{Mat}_N$ is equal to

$$m(a) = \sum_k \pm m_{i_1 \dots i_k} \text{tr}(t^{i_1} \dots t^{i_k})$$

with \pm derived from the Koszul sign rules.

We assume that the inner product is odd; then the Chern-Simons functional generates an odd vector field Q on $\Pi \mathcal{A}_N$; the conditions that should be imposed on the operations in A_∞ -algebra are equivalent to the condition $[Q, Q] = 0$ or, in other words, to the condition that Chern-Simons functional obeys the BV master equation.

Notice that two quasi-isomorphic A_∞ -algebras are physically equivalent (i.e. corresponding Chern-Simons functionals lead to the same physical results).

A differential associative algebra can be considered as an A_∞ -algebra where only operations $^{(1)}m$ and $^{(2)}m$ do not vanish; in this case both definitions of Chern-Simons functional coincide.

2 Observables of Chern-Simons theory.

Recall that in BV formalism a physical theory is specified by an action functional S defined on a space of fields considered as functions on odd symplectic manifold \mathcal{E} . A classical observable is defined as an even functional a obeying $\{S, a\} = 0$ or equivalently $\xi_S a = 0$ where S is the action functional obeying the master equation and ξ_S stands for the operator $\xi_S f = \{S, f\}$. (We restrict ourselves to the polynomial functionals.) Classical observables are related to infinitesimal deformations of the solution to the master equation. In what follows we will consider also odd functionals a obeying $\{S, a\} = 0$. These "odd observables" do not have a physical meaning of observables, but they correspond to odd infinitesimal deformations (deformations of the form $S + \epsilon a$ where ϵ is an odd parameter.) Trivial observables (observables of the form $a = \{S, b\}$) correspond to trivial deformations (deformations induced by infinitesimal change of variables). We will see that odd observables generate even symmetry transformations and even observables generate odd symmetry transformations.

Equivalently one can describe a theory by means of a homological vector field Q on \mathcal{E} preserving the odd symplectic form; then the first order differential operator \hat{Q} corresponding to Q can be represented in the form ξ_S . Notice that the set of observables depends only on Q (the symplectic form and the action functional are irrelevant). The field Q determines the equations of motion (a solution of equation of motion can be interpreted as a point in the zero locus of Q .) If there exists a Q -invariant symplectic form these equations of motion come from action functional.

If Chern-Simons theory is constructed by means of associative graded differential algebra \mathcal{A} with an odd inner product then every element of cyclic cohomology of \mathcal{A} specifies an observable [14]. This fact follows from the statement that infinitesimal deformations of \mathcal{A} into A_∞ -algebra with inner product are labelled by cyclic cohomology $HC(\mathcal{A})$ of \mathcal{A} [13]. Algebra \mathcal{A} determines Chern-Simons theory for all N ; the observables we are talking about are defined for every N .

The observables we consider can be constructed directly, without reference to [13]. By definition a cyclic cocycle on \mathcal{A} is a polylinear map $\sigma : \mathcal{A}^n \rightarrow \mathbb{C}$ satisfying some conditions. Such a map can be used to construct a polylinear map $\sigma_N : \mathcal{A}_N^n \rightarrow \mathbb{C}$ specifying a cyclic cocycle of \mathcal{A}_N . This follows from Morita invariance of cyclic cohomology ([10]). It is easy to give an explicit formula for σ_N , namely

$$\sigma_N(A_1, \dots, A_n) = \sum_{i_1, \dots, i_n} \sigma(a_1^{i_1, i_2}, a_2^{i_2, i_3}, \dots, a_n^{i_n, i_1}), A_k = (a_k^{i, j})$$

An observable corresponding to the cyclic cocycle is the functional $\sigma_N(A, \dots, A)$ defined on $\Pi\mathcal{A}_N$.

Obviously a product of observables is an observable, hence elements of $\text{Sym}\Pi HC(\mathcal{A}) = \sum \text{Sym}^k \Pi HC(\mathcal{A})$ specify observables defined for every N . Let us prove that all observables defined for every N are of this kind. To classify observables we should compute the cohomology of the first order differential operator $\hat{Q} = \xi_{CS}$ induced by the vector field Q in the space of polynomial functionals on \mathcal{A}_N . It is easy to check that up to parity reversion this operator can be identified with the differential in the definition of Lie algebra cohomology of $L_N\mathcal{A} = \text{Lie}\mathcal{A}_N$. It is well known [10] that for large N the cohomology of \mathcal{A} with trivial differential is isomorphic to $\text{Sym}[\Pi HC(\mathcal{A})]$.

For any dga \mathcal{A} there exists a homomorphism of $\text{Sym}[\Pi HC(\mathcal{A})]$ into the projective (inverse) limit of groups $H(L_N\mathcal{A})$; in the cases we are interested in this homomorphism is an isomorphism (we do not know whether this is true in general).

3 Ten-dimensional SUSY YM theory as generalized Chern-Simons theory.

We have constructed Chern-Simons theory starting with differential associative algebra \mathcal{A} equipped with closed trace tr that generates invariant inner product $\langle a, b \rangle = \text{tr}ab$. We have assumed that the inner product is non-degenerate; however, one can show that it suffices to assume the non-degeneracy of the induced inner product on homology. This remark

allows us to consider ten- dimensional SUSY YM theory and its dimensional reductions as generalized Chern-Simons theory.

We define Berkovits algebra B as the algebra of polynomial functions of pure spinor λ , odd spinor ψ and $x = (x^1, \dots, x^{10}) \in \mathbb{R}^{10}$. Sometimes it is convenient to modify this definition considering an algebra B^∞ consisting of functions that are polynomial in λ and ψ but smooth as functions of $x \in \mathbb{R}^{10}$. The differential is defined as the derivation

$$d = \lambda^\alpha \left(\frac{\partial}{\partial \psi^\alpha} + \Gamma_{\alpha\beta}^i \psi^\beta \frac{\partial}{\partial x^i} \right). \quad (2)$$

The algebra B_d (Berkovits algebra reduced to d -dimensional space) is the algebra of functions depending on pure spinor λ , odd spinor ψ and $x = (x^1, \dots, x^d) \in \mathbb{R}^d$. The differential is defined by the same formula.

One can use (1) to define an odd vector field Q on $B_d \otimes Mat_N$. It is well known ([2], [6],[3]) that for every solution $A_\alpha(x, \theta)$ of equations of motion of ten-dimensional SUSY YM one can construct a point $\lambda^\alpha A_\alpha(x, \theta) \in B_{10} \otimes Mat_N$ belonging to the zero locus of Q . (We are working in superspace formalism, the superfield A_α takes values in $N \times N$ matrices.) Starting with a solution of equations of motion reduced to d -dimensional space we obtain a point of the zero locus of the vector field Q on $B_d \otimes Mat_N$.

For $d = 0$ we obtain the reduced Berkovits algebra related to ten-dimensional SUSY YM theory reduced to a point. Introducing an appropriate trace we can apply the techniques of generalized Chern-Simons theory to this algebra. To apply these techniques to the case $d > 0$ we should introduce the notions of local observables and local trace (see the next section).

4 Lie algebra of local observables in the classical BV formalism

Let $f : M \rightarrow N$ be a \mathbb{C} -linear map of graded R -modules where R is a graded commutative algebra over \mathbb{C} . We define $[x, f] : M \rightarrow N$ by the formula $xf(m) - (-1)^{|f||x|}f(xm), x \in R$.

The map f is local if $[x_1, \dots, [x_n, f] \dots] \equiv 0$ for some n and all $x_1, \dots, x_n \in R$. If R is an algebra of smooth functions on $\mathbb{R}^{n|k}$ and M, N are the space of sections of finite rank vector bundles then f defines a differential operator between vector bundles.

If R, M, N are differential graded objects then the definition can be weakened by replacing $[x_1, \dots, [x_n, f] \dots] \equiv 0$ by $[x_1, \dots, [x_n, f] \dots] \equiv [d, g]$ for some $g : M \rightarrow N$.

Let A be an associative graded algebra over R and M is a graded A -bimodule. The graded space of Hochschild cochains $C^k(A, M) = \text{Hom}_{\mathbb{C}}(A^{\otimes k}, M)$ contains a subspace $C_{loc}^k(A, M)$ of cochains \mathcal{O} that are local with respect to all variables (i.e., $f(x) = \mathcal{O}(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_k)$ is a local map of R -bimodules for $i = 1, \dots, k$). We omit the standard definition of the Hochschild differential referring to [1] or to [9]. By definition multiplication and the differentials in A and M are local. This is why $\prod_k C_{loc}^k(A, M)$ is a subcomplex in $\prod_k C^k(A, M)$. The cohomology of $\prod_k C_{loc}^k(A, M)$ is denoted by $\text{HH}_{loc}^k(A, M)$. When no confusion is possible we will drop the R -dependence in the cohomology: $\text{HH}_{loc}^k(A, M) = \text{HH}_{loc}^k(A, M)$

When $A = R$ is an algebra of functions on a smooth or an affine algebraic manifold X and M is space of sections of a vector bundle over X then $\text{HH}_{loc}^k(A, M) = \text{HH}^k(A, M)$ (c.f. [7], Section 4.6.1.1.)

In the following we will assume that R is an algebra of functions on a smooth or an algebraic supermanifold X . This allows us to define the complex of integral forms $\{\Omega_R^{-i}\}$, equipped with the de Rham differential d_{dr} .

By definition a local trace is a series of graded local maps $\text{tr}_i : A \rightarrow \Omega_R^{-i}$ that for $i \geq 0$ satisfy

$$\text{tr}_i(d_A a) = -d_{dr} \text{tr}_{i+1}(a), \text{tr}_i([a, b]) = 0.$$

When $R = \mathbb{C}$ this becomes a definition of an ordinary trace that determines an inner product $\text{tr}(ab)$ on A . This inner product allows us to define a map of cohomology groups $\text{HH}^i(A, A) \rightarrow \text{HH}^i(A, A^*)$. Our next set of definitions is intended to create a setting which would accommodate a similar homomorphism in a local setting.

The complex $C^k(A, A^*)$ coincides with the space of graded maps $\text{Hom}(A^{\otimes(k+1)}, \mathbb{C})$.

The complex is equipped with the differential d_{hoch} (see [1], [9] for details) that computes Hochschild cohomology $\mathrm{HH}^i(A, A^*)$. By definition the complex $C_{loc}^k(A)$ is a subcomplex of the complex $\prod_{ij} \mathrm{Hom}_{\mathbb{C}}(A^{\otimes k+1}, \Omega_R^{-i})$ of R -local maps. The differential d is equal to the sum $d_{hoch} + d_{dr}$, where d_{dr} is acting in $\bigoplus_i \Omega_R^{-i}$. We denote the cohomology of $C_{loc}^k(A)$ by $\mathrm{HH}_{loc}^k(A)$. Note that when $R = \mathbb{C}$ the cohomology $\mathrm{HH}_{loc}^k(A)$ becomes equal to $\mathrm{HH}^k(A, A^*)$.

The R -local cyclic cohomology are defined along the same line: the subspace $\prod_k CC^k(A)$ of $C_{loc}^k(A)$ consisting of cyclic cochains is a subcomplex. Corresponding cohomology is denoted by $\mathrm{HC}_{loc}^n(A)$.

The groups $\mathrm{HH}_{loc}^n(A, A)$, $\mathrm{HH}_{loc}^n(A)$ and $\mathrm{HC}_{loc}^n(A)$ have "multi-trace" generalizations. By definition the chains of $\mathrm{HH}_{loc,mt}^n(A, A)$ is a subspace of the R -local maps in $\mathrm{Hom}_{\mathbb{C}}(T(A) \otimes \mathrm{Sym}[\Pi CC(A)], A)$. The space of chains for $\mathrm{HH}_{loc,mt}^n(A)$ is a subspace of local maps in $\mathrm{Hom}_{\mathbb{C}}(T(A) \otimes A \otimes \mathrm{Sym}[\Pi CC(A)], \Omega_R)$, the space of chains for $\mathrm{HC}_{loc,mt}^n(A)$ is a subspace of local maps in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}[\Pi CC(A)], \Omega_R)$. Keep in mind that such generalizations for the classical nonlocal theories reduce to the space of linear maps of $\bigoplus_k \mathrm{HH}^k(A, A)$ and $\bigoplus_k \mathrm{HH}^k(A, A^*)$ with $\mathrm{Sym}[\Pi \mathrm{HC}(A)]$.

The local version of the cyclic and Hochschild cohomology shares many common properties with their classical counterparts. In particular there is a long exact sequence

$$\cdots \rightarrow \mathrm{HH}_{loc}^n(A) \rightarrow \mathrm{HC}_{loc}^{n-1}(A) \rightarrow \mathrm{HC}_{loc}^{n+1}(A) \rightarrow \mathrm{HH}_{loc}^{n+1}(A) \rightarrow \cdots$$

There is a similar sequence for the multi-trace version of the theories.

A local trace functional $\{\mathrm{tr}_I\}$, which we defined above, is a cocycle in $\mathrm{HC}_{loc}^0(A)$. It defines maps $\mathrm{HH}_{loc}^s(A, A) \rightarrow \mathrm{HH}_{loc}^s(A)$, $\mathrm{HH}_{loc,mt}^s(A, A) \rightarrow \mathrm{HH}_{loc,mt}^s(A)$ by the formula

$$c(a_1, \dots, a_s) \rightarrow \sum_i \mathrm{tr}_i(c(a_1, \dots, a_s) a_{s+1}).$$

We say that the trace $\mathrm{tr} = \{\mathrm{tr}_i\}$ is homologically nondegenerate if it induces an isomorphism $\mathrm{HH}_{loc}^s(A, A) \rightarrow \mathrm{HH}_{loc}^s(A)$.

Some of the discussion from Section 2 can be carried out in the local setting. In

particular local vector fields, local functionals and local k-forms obviously make sense. There are homomorphisms

$$\mathrm{HH}_{loc,mt}^n(A, A) \rightarrow \mathrm{H}^n(L_N A, L_N A),$$

$$\mathrm{HC}_{loc,mt}^n(A) \rightarrow \mathrm{H}^n(L_N A),$$

$$\mathrm{HH}_{loc,mt}^n(A) \rightarrow \mathrm{H}^n(L_N A, L_N A^*)$$

The relative Hochschild cohomology group $\mathrm{HH}_R^n(A, A)$ maps to $\mathrm{HH}_{loc}^n(A, A)$. The relations of the relative groups $\mathrm{HH}_R^n(A, A^*)$ and $\mathrm{HC}_R^n(A)$ to $\mathrm{HH}_{loc}^n(A)$ and $\mathrm{HC}_{loc}^n(A)$ is obscure.

The last type of generalization is intended for a dga A without a unit over a commutative unital dga R . The definitions follow closely the above outline. The reader can consult on the details of cohomology theory of algebras without a unit in [9].

5 Symmetry preserving deformations

As we mentioned already infinitesimal deformations of the solution of master equation correspond to observables. Observables belonging to the same cohomology class specify equivalent deformations, i.e. deformations related by a change of variable (by a field redefinition). This means that in BV formalism the deformations of physical theory with action functional S are labeled by the cohomology of the differential ξ_S .

As we have seen an even element of $\Lambda HC(\mathcal{A})$ specifies an observable of Chern-Simons theory defined for every N ; hence it determines an infinitesimal deformation of Chern-Simons theory defined for all N .

We will be interested in the deformations of Chern-Simons theory that are defined for every N and preserve the symmetry of original theory.

Let us make some general remarks about symmetries in BV formalism. If the equations of motion are specified by a homological vector field Q then every vector field q commuting with Q determines a symmetry of equations of motion. (The vector field q is tangent to

the zero locus of Q .) The vector field q can be even or odd; in other words we can talk about super Lie algebra of symmetries. However, among these symmetries there are trivial symmetries, specified by vector fields of the form $[Q, a]$ where a is an arbitrary vector field. This means that the super Lie algebra \mathcal{L} of non-trivial symmetries of EM can be described as homology of the space of all vector fields with respect to the differential defined as a commutator with Q . If we would like to consider only Lagrangian symmetries, i.e. symmetries corresponding to vector fields q having the form ξ_s we obtain a Lie algebra $\tilde{\mathcal{L}}$ isomorphic to the homology of operator \hat{Q} . Notice, that both \mathcal{L} and $\tilde{\mathcal{L}}$ depend only on the vector field Q . However, the natural homomorphism $\tilde{\mathcal{L}} \rightarrow \mathcal{L}$ does depend on the choice of odd symplectic structure on \mathcal{E} .

Let us calculate \mathcal{L} and $\tilde{\mathcal{L}}$ in the case of Chern-Simons theory restricting ourselves to the symmetry transformations, that can be applied for all N . The calculation of $\tilde{\mathcal{L}}$ coincides with calculation of observables (up to parity reversal); we obtain $\tilde{\mathcal{L}} = \Lambda HC(\mathcal{A})$.

To calculate \mathcal{L} we should study the cohomology $H(L_N(\mathcal{A}), L_N(\mathcal{A}))$ (the cohomology of the Lie algebra $L_N(\mathcal{A})$ with the coefficients in adjoint representation). In the case of algebra \mathcal{A} with zero differential these cohomology are well known for large N ([5], [10]) ; they are equal to

$$\text{Sym}[\text{IHC}(\mathcal{A})] \otimes \text{HH}(\mathcal{A}, \mathcal{A}).$$

For dga algebras we have maps

$$H(L_N \mathcal{A}, L_N \mathcal{A}) \leftarrow \text{Sym}[\text{IHC}(\mathcal{A})] \otimes \text{HH}(\mathcal{A}, \mathcal{A})$$

In interesting cases the LHS stabilizes for large N and becomes isomorphic to the RHS. It is not clear whether this situation is general.

We can say that \mathfrak{g} is a symmetry Lie algebra if it is embedded into \mathcal{L} (or into $\tilde{\mathcal{L}}$ if we would like to consider only Lagrangian symmetries). Fixing some system of generators e_α in \mathfrak{g} with structure constants $f_{\alpha\beta}^\gamma$ one can say that \mathfrak{g} is a symmetry Lie algebra of BV theory with homological vector field Q if there exist symmetry transformations q_α satisfying

commutation relations

$$[q_\alpha, q_\beta] = f_{\alpha\beta}^\gamma q_\gamma + [Q, q_{\alpha\beta}] \quad (3)$$

for some vector fields $q_{\alpha\beta}$. However, it is useful to accept a more restrictive definition of symmetry Lie algebra. We will say that \mathfrak{g} is a symmetry algebra if we have an L_∞ -homomorphism of \mathfrak{g} into the differential Lie algebra of vector fields (this Lie algebra is equipped with a differential defined as a commutator with Q). L_∞ -homomorphism of Lie algebra \mathfrak{g} with generators e_α into differential Lie algebra \mathcal{V} is defined as a sequence $q_\alpha, q_{\alpha_1, \alpha_2}, \dots \in \mathcal{V}$ obeying some relations, generalizing (3). (See [8] for details.)

Let us suppose that Lie algebra \mathfrak{g} acts on the differential algebra \mathcal{A} . This means that we have fixed a homomorphism $\phi : \mathfrak{g} \rightarrow \text{Der} \mathcal{A}$ of \mathfrak{g} into the Lie algebra of derivations of \mathcal{A} . (It is sufficient to assume that we have an L_∞ action, i.e. an L_∞ homomorphism of \mathfrak{g} into differential algebra $\text{Der}(\mathcal{A})$.) This action specifies \mathfrak{g} as a Lie algebra of symmetries of Chern-Simons functional for every N . We are interested in infinitesimal deformations of this functional preserving these symmetries (\mathfrak{g} -invariant deformations). We identify two deformations related by the change of variables.

The space $C^\bullet(\mathcal{A}, \mathcal{A})$ of Hochschild cochains with coefficients in \mathcal{A} has a natural L_∞ action of \mathfrak{g} , hence we can consider the cohomology of Lie algebra \mathfrak{g} with coefficients in this module. We will denote this cohomology by $HH_\mathfrak{g}^\bullet(\mathcal{A}, \mathcal{A})$ and call it Lie -Hochschild cohomology with coefficients in \mathcal{A} . (For trivial \mathfrak{g} it coincides with Hochschild cohomology of \mathcal{A} , for trivial \mathcal{A} with Lie algebra cohomology of \mathfrak{g} .) Similarly we can define $HC_\mathfrak{g}^\bullet(\mathcal{A})$ (Lie-cyclic cohomology), $HH_\mathfrak{g}^\bullet(\mathcal{A}, \mathcal{A}^*)$. There are also multi-trace version of these groups. For example the multi-trace version of $HC_\mathfrak{g}^\bullet(\mathcal{A})$ uses the symmetric algebra of standard cyclic bicomplex $CC^i(\mathcal{A})$. The multi-trace cyclic cohomology group $HC_{\mathfrak{g}, mt}^\bullet(\mathcal{A})$ is the cohomology of the bicomplex $C^\bullet(\mathfrak{g}, \text{Sym}[CC^\bullet(\mathcal{A})])$. The multi-trace equivariant version of Hochschild cohomology is cohomology of the tri-complex $C^\bullet(\mathfrak{g}, \text{Sym}[CC^\bullet(\mathcal{A})] \otimes C^\bullet(\mathcal{A}, \mathcal{A}))$. One can prove the following theorem:

The \mathfrak{g} -invariant deformations of Chern-Simons action functional $CS(\mathcal{A})$ that are defined

for all N simultaneously are labelled by the elements of $\mathrm{HC}_{\mathfrak{g},mt}^\bullet(\mathcal{A})$.

We can use this theorem to study supersymmetric deformations of ten-dimensional SUSY YM theory represented as Chern-Simons theory corresponding to the Berkovits algebra.

There is a number of modifications of the Berkovits algebra B_{10} that depend on smoothness of its elements as functions on \mathbb{R}^{10} and their asymptotics at infinity. Possible choices are polynomials functions, which are elements of $\mathbb{C}[x_1, \dots, x_{10}]$. This way we get B_{10}^{poly} . Similarly we can get an analytic modification B_{10}^{an} which contains the algebra of analytic functions $C^{an}(\mathbb{R}^{10})$ or the smooth version $B_{10}^\infty \supset C^\infty(\mathbb{R}^{10})$, with or without restriction on asymptotics at infinity. Our following computations don't depend on what pair of algebras

$$R = \mathbb{C}[x_1, \dots, x_{10}] \subset B_{10}^{poly} = A$$

$$R = C^{an}(\mathbb{R}^{10}) \subset B_{10}^{an} = A \text{ or}$$

$$R = C^\infty(\mathbb{R}^{10}) \subset B_{10}^\infty = A.$$

we choose for cohomology computations. This is why we will use B_{10} as a unifying notation for all modifications.

We can calculate groups $\mathrm{HH}_{loc \ \mathfrak{su}(5)}^i(B_{10}, B_{10})$. These linear spaces have an additional conformal grading by eigenvalues of the dilation operator scaled by the factor of two²: $\mathrm{HH}_{loc \ \mathfrak{su}(5)}^i(B_{10}, B_{10}) = \bigoplus_{k \in \mathbb{Z}} \mathrm{HH}_{loc \ \mathfrak{su}(5)}^{i,k}(B_{10}, B_{10})$.

They can be expressed in terms of the groups $H^{s,t}(L, U(TYM))$ considered in [11]:

$$\mathrm{HH}_{loc \ \mathfrak{su}(5)}^{i,k}(B_{10}, B_{10}) = H^{i+k,i}(L, U(TYM))$$

The groups $H^{k,t}(L, U(TYM))$ were calculated in [11] for $k = 2$. (See also [12].) Similar methods can be applied for other values of k .

²We do this to avoid fractional gradation in spinor components.

6 Construction of deformations

One can construct some interesting symmetry preserving deformations starting with homology classes of symmetry Lie algebra \mathfrak{g} .

The application of the homology of \mathfrak{g} to the analysis of deformations is based on the construction of the homomorphism

$$\psi : H_i(\mathfrak{g}, N) \rightarrow H^{s-i}(\mathfrak{g}, N) \quad (4)$$

for arbitrary differential module N with L_∞ action of \mathfrak{g} . Here s stands for the number of even generators of \mathfrak{g} . This homomorphism is described in the appendix.

We will apply the homomorphism ψ to the construction of symmetry preserving deformations.

Let \mathcal{A} be a differential \mathbb{Z} -graded associative algebra.³ Let us assume that A is equipped with L_∞ action of Lie algebra \mathfrak{g} . Then $C^\bullet(A, A^*)$ and $HH^\bullet(A, A^*)$ are $L_\infty \mathfrak{g}$ -modules and we can talk about the homology and cohomology of \mathfrak{g} with coefficients in these modules. (Recall that $C^k(A, A^*)$ stands for the module of Hochschild co-chains, i.e. of k -linear functionals on A with values in A^* . Notice, that these co-chains can be identified with $(k+1)$ -linear functionals with values in \mathbb{C} .) The complex $C^\bullet(A, A^*)$ has an additional operation of degree minus one :

$$C^\bullet(A, A^*) \rightarrow C^{\bullet-1}(A, A^*)$$

-the Connes differential B . The map B is a composition of two maps αB_0 , which look particularly simple if the degrees of all elements are even. The operator α is the operator of cyclic antisymmetrisation. The operator B_0 is defined by the formula

$$(B_0\psi)(a^0, \dots, a^n) = \psi(1, a^0, \dots, a^n) - (-1)^{n+1}\psi(a^0, \dots, a^n, 1)$$

³ One can generalize our constructions to the case of A_∞ algebras using the fact that a \mathbb{Z} -graded A_∞ algebra is quasi-isomorphic to differential graded algebra,

The reader may consult [4], [9] for details.⁴

This operator induces map on $C_{\mathfrak{g}}^i(A, A^*)$ and $C^{\mathfrak{g}i}(A, A^*)$, denoted by the same symbol; it anticommutes with $d_{\mathfrak{g}}$ and d_c .

The Connes operator induces a differential on Hochschild cohomology. Let us assume in that the cohomology of B in $H^i(A, A^*)$ is trivial for $i > 0$ and is one-dimensional for $i = 0$.⁵ This assumption permits us to construct an element of homology $H_{\bullet}(\mathfrak{g}, C^{\bullet}(A, A^*))$ starting with any element $c_0 \in H_{\bullet}(\mathfrak{g}, C^{\bullet}(A, A^*))$ obeying $Bd_{\mathfrak{g}}c_0 = 0$. The construction is based on the observation that due to triviality of the cohomology of B we can represent $d_{\mathfrak{g}}c_0$ as Bc_1 . Applying $d_{\mathfrak{g}}$ to both parts of equation $d_{\mathfrak{g}}c_0 = Bc_1$ we obtain $Bd_{\mathfrak{g}}c_1 = 0$; this equality allows us to continue the process. The process will terminate when $d_{\mathfrak{g}}c_i = 0$. This must happen for some i because $d_{\mathfrak{g}}$ decreases the degree in $\text{Sym}(\Pi\mathfrak{g})$ (the number of ghosts). The element c_i specifies the homology class we are interested in.

Let us describe the construction of elements of $H_{\bullet}(\mathfrak{g}, C^{\bullet}(A, A))$, which uses homology classes of super Lie algebra \mathfrak{g} with trivial coefficients and a \mathfrak{g} -equivariant trace tr as an input. (We assume that the trace specifies non-degenerate inner product on cohomology.) The trace determines a \mathfrak{g} -equivariant map $A^* \rightarrow A$ and therefore a homomorphism $H_{\bullet}(\mathfrak{g}, C^{\bullet}(A, A^*)) \rightarrow H_{\bullet}(\mathfrak{g}, C^{\bullet}(A, A))$. This means that it is sufficient to construct an element of $H_{\bullet}(\mathfrak{g}, C^{\bullet}(A, A^*))$.

Let us take a representative $c \in \text{Sym}(\Pi\mathfrak{g})$ of homology class of the Lie algebra \mathfrak{g} . Then we can define c_0 by the formula $c_0 = c \otimes \epsilon$, where ϵ stands for a homomorphism of the algebra A into a field.⁶ It is easy to check that $B\epsilon = 0$ and that ϵ specifies a non-trivial class in the homology of B . We see that $Bd_{\mathfrak{g}}c_0 = -d_{\mathfrak{g}}Bc_0 = 0$. This means that we can apply the iterative construction described above to obtain a cycle c_l . The corresponding

⁴The operator B transforms Hochschild n -cochain into cyclic $(n-1)$ -cochain; it generates a homomorphism $HH^n(A, A^*) \rightarrow HC^{n-1}(A)$ in Connes exact sequence.

⁵ This is true, for example, if there exists an auxiliary grading by means of non-negative integers with one-dimensional grading zero component.

⁶ If A is represented as a direct sum of one-dimensional subalgebra generated by the unit and ideal I (augmentation ideal) then ϵ is a projection of A on the first summand.

homology class $[c_l] \in H_\bullet(\mathfrak{g}, C^\bullet(A, A^*))$ is the class we need.

Remark Recall that we have defined \mathfrak{g} -equivariant version of cyclic cohomology (Lie-cyclic cohomology) $HC_\mathfrak{g}^\bullet(\mathcal{A})$ as cohomology of the Lie algebra \mathfrak{g} with coefficients in cyclic cochains considered as a differential \mathfrak{g} -module. It is rather straightforward to construct Connes long exact sequence. The main corollary of this construction is that classes c_l are images of classes in cyclic cohomology.

Remark The equivariant version of the package $HC^\bullet(\mathcal{A})$, $HH^\bullet(\mathcal{A}, \mathcal{A})$, $HH^\bullet(\mathcal{A})$ makes sense for both of local versions described in Section 4.

7 Appendix. Homology of super Lie algebras

7.1 Finite-dimensional super Lie algebras

Let us consider first the cohomology of finite-dimensional super Lie algebras. This cohomology is defined in terms of a differential

$$d = \frac{1}{2}(-1)^{|b^k|} f_{lk}^m b^k b^l c_m$$

where f_{kl}^m are structure constants of super Lie algebra G in some basis t_k . The operators b^k and c_k correspond to elements of basis, but have parity opposite to the parity of elements of basis. They satisfy canonical (anti)commutation relations: $[c_k, b^l] = \delta_k^l$; in other words they can be considered as generators of super Weyl algebra W_{rs} where r stands for the number of even generators and s stands for the number of odd generators. The differential acts in any representation of Weyl algebra (in any W_{rs} -module); the cohomology can be defined by means of any representation and depends on the choice of representation. We will assume that the representation \mathcal{F} of Weyl algebra W_{rs} is graded in such a way that b^k raises grading by 1 and c_k decreases grading by 1, then the differential increases grading by 1. The cohomology is also graded in this case. One can define cohomology of super Lie algebra G with coefficients in G -module N by means of the differential $d + \sum T_k b^k$ on the space $\mathcal{F} \otimes N$ (Here T_k denotes the action of generator $t_k \in G$ on N .) We will

use the notation $H^k(G, N|\mathcal{F})$ for k -dimensional cohomology of super Lie algebra G with coefficients in G -module N calculated by means of W_{rs} -module \mathcal{F} .

If G is a conventional Lie algebra then $r = 0$, the super Weyl algebra is a Clifford algebra with $2s$ generators. Irreducible representation in this case is unique; the representation space can be realized as Grassmann algebra (as algebra of functions of s anticommuting variables) where c_k and b^k act as derivatives and multiplication operators. We come to the standard notion of cohomology of Lie algebra. (However, in the case when G is an infinite-dimensional Lie algebra the irreducible representation of corresponding Clifford algebra is not unique; this remark leads to the notion of semi-infinite cohomology.) If $r > 0$ the super Weyl algebra is a tensor product of Weyl algebra W_r and Clifford algebra Cl_s ; further W_r is a tensor product of r copies of Weyl algebra $W = W_1$.⁷

Let us consider first of all representations of the algebra W having generators b, c with relation $[c, b] = 1$. The simplest of these representations F_+ is realized in the space of polynomials $\mathbb{C}[t]$ where c acts as a derivation and b as a multiplication by t . The grading is given by the degree of polynomial. This representation can be described also as representation with cyclic vector Φ obeying $c\Phi = 0$ (Fock representation with vacuum vector Φ .) Another representation F_- can be constructed as a representation with cyclic vector Ψ obeying $b\Psi = 0$, $\deg \Psi = 0$. To relate these two representations we consider the representation F in the space $\mathbb{C}[t, t^{-1}]$ (polynomials of t and t^{-1}). The operators c and b again act as derivation and multiplication by t . It is easy to check that factorizing F with respect to subrepresentation F_+ we obtain a representation isomorphic to F_- (the polynomial t^{-1} plays the role of cyclic vector Ψ). Notice, however, that the grading in F_- does not coincide with the grading in F/F_+ (the degree of t^{-1} is equal to -1). One can say that as graded module F/F_+ is isomorphic to $F_-[-1]$ (to F_- with shifted grading).

Let us represent W_{rs} as a tensor product $W \otimes W_{r-1, s}$. For every representation E of

⁷Notice that we work in algebraic setting. It is well known that for correct definition of unitary representation Weyl algebra with finite number of generators has only one irreducible unitary representation. This statement cannot be applied to representations at hand.

second factor we can construct two representations of W_{rs} as tensor products $F_+ \otimes E$ and $F_- \otimes E$. The relation $F/F_+ = F_-[-1]$ permits us to construct a map

$$H^k(G, N|F_- \otimes E) \rightarrow H^k(G, N|F_+ \otimes E) \quad (5)$$

This map is analogous to picture changing operator in BRST cohomology of superstring. It can be regarded as coboundary operator in exact cohomology sequence corresponding to short exact sequence

$$0 \rightarrow F_+ \otimes E \rightarrow F \otimes E \rightarrow F_- \otimes E \rightarrow 0.$$

Notice that coboundary operator raises degree by 1, but taking into account the the shift of grading in F_- we see that (5) does not change the degree.

We will consider irreducible representations $F_{\epsilon_1, \dots, \epsilon_r}$ of W_{rs} defined as tensor product of representations F_{ϵ_k} and irreducible representation of Clifford algebra. (Here $\epsilon_k = \pm$.) These representations can be defined also as Fock representations with vacuum vector Φ obeying $c_k \Phi = 0$ if $\epsilon_k = +$ and $b^k \Phi = 0$ if $\epsilon_k = -$. The grading is determined by the condition $\deg \Phi = 0$. The cohomology corresponding to representation with all $\epsilon_k = +$ coincides with standard cohomology of super Lie algebra:

$$H^k(G, N|F_{+, \dots, +}) = H^k(G, N).$$

The cohomology corresponding to representation with all $\epsilon_k = -$ is closely related to homology of super Lie algebra. If $f_{kl}^l = 0$ we have

$$H^k(G, N|F_{-, \dots, -}) = H_{s-k}(G, N). \quad (6)$$

To check (6) we notice that homology can be defined by means of differential

$$\partial = \frac{1}{2}(-1)^{|\gamma_k|} f_{lk}^m \gamma_m \frac{\partial^2}{\partial \gamma_k \partial \gamma_l} + T_k \frac{\partial}{\partial \gamma_k}$$

acting in the space of polynomial functions of variables γ_k (ghost variables). (Here as earlier T_k denotes the action of $t_k \in G$ on G -module N . The ghost variables have the

parity opposite to the parity of t_k .) We can rewrite this differential in the form

$$\partial = \frac{1}{2}(-1)^{|b_k|} f_{lk}^m c_m b^k b^l + T_k b^k$$

where c_k, b^l satisfy canonical (anti)commutation relations. If $f_{kl}^l = 0$ the differential takes the form of cohomology differential acting in the space $F_{-,\dots,-}$. (The constant polynomial is a cyclic vector Φ obeying $b_k \Phi = 0$.) However, the grading is different: in the space of polynomial functions of γ_k the operator c_k increases degree by 1 (instead of decreasing it by 1 in cohomological grading). The grading of the cyclic vector Φ (of the Fock vacuum) is also different in homological and cohomological setting (0 versus s). We obtain the formula (6).

Applying r times the homomorphism (5) we obtain a homomorphism from homology into cohomology. More precisely, if $f_{kl}^l = 0$ we obtain a homomorphism

$$H_i(G, N) \rightarrow H^{s-i}(G, N)$$

.

7.2 Infinite-dimensional super Lie algebras

Recall that the cohomology of finite-dimensional super Lie algebra G with coefficients in G -module N were defined by means of differential

$$d = \frac{1}{2}(-1)^{|b_k|} f_{lk}^m b^k b^l c_m + T_k b^k \quad (7)$$

acting on the tensor product $\mathcal{F} \otimes N$. Here \mathcal{F} is a representation of super Weyl algebra with generators b^k, c_l , the symbol f_{kl}^m denotes structure constants of G in the basis t_k and T_k stands for the operator in N corresponding to t_k . The space \mathcal{F} can be considered as a G -module; the elements $t_k \in G$ act as operators

$$\tau_k = f_{kl}^m b^l c_m. \quad (8)$$

These operators obey relations

$$[\tau_k, \tau_l] = f_{kl}^m \tau_m, \quad (9)$$

$$[\tau_k, b^m] = f_{kl}^m b^l, \quad (10)$$

$$[\tau_k, c_l] = f_{kl}^m c_m. \quad (11)$$

The differential d obeys

$$[d, b^m] = \frac{1}{2} f_{kl}^m b^k b^l, \quad (12)$$

$$[d, c_l] = \tau_l + T_l. \quad (13)$$

Let us consider now the case when G is an infinite-dimensional super Lie algebra and N is a projective representation of G (=a module over central extension of G). We will keep the notation t_k for the elements of basis of G and f_{kl}^m for structure constants. *We will assume that for fixed indices k, l there exists only finite number of indices m such that $f_{kl}^m \neq 0$. Similarly, if indices k, m are fixed then $f_{kl}^m \neq 0$ only for finite number of indices l .* The formulas (7) and (8) in general do not make sense in this situation. However, the RHS of (10) and (11) is well defined. We will assume \mathcal{F} is an irreducible representation of Weyl algebra; then these formulas specify τ_k uniquely up to an additive constant. If the solution for τ_k does exist it specifies a projective representation of G :

$$[\tau_k, \tau_l] = f_{kl}^m \tau_m + \gamma_{kl}.$$

The constants γ_{kl} determine a two-dimensional cocycle of G ; in physics it is related to central charge.

We assume that the two-dimensional cohomology class of G corresponding to the projective module N is opposite to the cohomology class of γ . This means that for appropriate choice τ_l the expression $\tau_l + T_l$ (the RHS of (13)) specifies a genuine representation of G

We will consider the case when $\mathcal{F} = \mathbb{F}_I$ is a Fock module (a module with a cyclic vector Φ obeying $b^k \Phi = 0$ for $k \in I$, $c_l \Phi = 0$ if $l \in J$ where J denotes the complement to

I). Here I stands for some set of indices; we assume that there exists only a finite set of triples (k, l, m) with $f_{kl}^m \neq 0$ obeying $k, l \in J, m \in I$.⁸ Then τ_k obeying equations (10) and (11) can be written in terms of normal product

$$\tau_k = f_{kl}^m : b^l c_m : . \quad (14)$$

Under our assumptions the RHS of (12) and (13) specifies a well defined operator on \mathcal{F} . Considering these formulas as equations for d we see that they determine d up to an additive constant. Requiring $d^2 = 0$ we obtain the following expression for d :

$$d = \frac{1}{2}(-1)^{|\gamma_k|} f_{lk}^m : b^k b^l c_m : + T_k b^k. \quad (15)$$

One defines the cohomology of G with coefficients in N by means of differential d . The cohomology in general depends on the choice of set I (on the choice of picture). One can introduce grading in \mathcal{F} assuming that b^k increases degree by 1, c_l decreases degree by 1 and that $\deg \Phi = 0$. Using this grading (and grading in N) one can define grading on cohomology. We will use the notation $H^n(G, N; I)$ for k -dimensional cohomology.

We would like to study relation between $H^n(G, N; I)$ and $H^n(G, N; I')$ (the dependence of cohomology on the choice of the picture). We will analyze the case when I' is obtained from I by deleting one index k . Let us notice first of all that in the case when t_k is an even generator (corresponding ghosts are odd) $H^n(G, N; I) = H^n(G, N; I')$; this follows from the fact that \mathbb{F}_I can be identified with \mathcal{F}' . (If a vector Φ obeys $b^k \Phi = 0$ for $k \in I$, $c_l \Phi = 0$ for $l \in J$ the vector $\Phi' = c_k \Phi \in \mathcal{F}_I$ obeys $b^k \Phi' = 0$ for $k \in I'$, $c_l \Phi = 0$ if $l \in J'$ where J' stands for the complement of I' .) If the generator t_k is odd (corresponding ghosts are even) then repeating the arguments used for finite-dimensional Lie algebras we can construct a homomorphism

$$H^n(G, N; I) \rightarrow H^n(G, n; I') \quad (16)$$

⁸In many interesting situations G as a vector space can be represented as a direct sum of two subalgebras; the representation of the set of indices as a disjoint union of I and J is related to this decomposition. In this case the cohomology we are interested in is called semi-infinite cohomology.

(picture changing operator).

This homomorphism is not an isomorphism in general. However, it is an isomorphism in cases relevant for string theory (when G is a superanalog of Virasoro algebra).

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